

# GLOBAL NILPOTENT CONE IS ISOTROPIC: PARAHORIC TORSORS ON CURVES

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ABSTRACT. In this note we show that the global nilpotent cone is an isotropic substack of the cotangent bundle of the moduli stack of parahoric torsors on a smooth projective curve. Further for the case of symplectic parabolic bundles, we show that the global nilpotent cone is infact a Lagrangian complete intersection substack.

## 1. INTRODUCTION

Let  $X$  be a smooth projective curve over  $\mathbb{C}$  of genus  $g > 1$  and  $G$  a complex semisimple group. Denote by  $Bun_G$ , the moduli stack of principal  $G$ -bundles on  $X$  and by  $\mathbb{P}$ , the universal bundle in  $X \times Bun_G$ . Let  $T^*Bun_G = Spec(Sym^*(ad(\mathbb{P})))$  be the cotangent stack of  $Bun_G$ . For a scheme  $S$  and a principal bundle  $E$  on  $X \times S$ , it is easy to see that [see [4]]

$$T_E^*Bun_G = H^0(X \times S, ad(E) \otimes \Omega_{X \times S/S}^1)$$

Now for a point  $y \in X \times S$ , and  $s \in H^0(X \times S, ad(E) \otimes \Omega_{X \times S/S}^1)$ , we have  $s(y) \in ad(E)(y) \otimes \Omega_{X \times S/S}^1(y) \simeq \mathfrak{g} \otimes k(y)$ . Thus it makes sense to call  $s$  nilpotent if for any point  $y \in X \times Y$ , we have  $s(y)$  is a nilpotent element in  $\mathfrak{g} \otimes k(y)$ . Following Laumon([8]) and Ginzburg([4]), the *global nilpotent cone* of  $T^*Bun_G$  is defined as the substack  $\mathcal{N}ilp$ , whose  $S$ -valued points are

$$\mathcal{N}ilp(S) = \{(P, s) \mid P \in Bun_G(S), s \in H^0(X \times S, ad(P) \otimes \Omega_{X \times S/S}^1) \text{ is nilpotent}\}.$$

An equivalent definition of the nilpotent substack  $\mathcal{N}ilp$ , is using the Hitchin morphism. Let  $\{F_1, \dots, F_l\}$  be a minimal set of homogeneous generators for the algebra  $\mathbb{C}[\mathfrak{g}]^G$ . Let  $d_i$  be the degree of  $F_i$  and denote by  $H$ , the affine space

$$H = \bigoplus_{i=1}^l H^0(X, K_X^{d_i})$$

Then  $\dim(H) = \dim(Bun_G)$  and Hitchin defines a morphism

$$\chi : T^*Bun_G \rightarrow H$$

where for a scheme  $S$ ,  $P \in Bun_G(S)$  and  $s \in H^0(X \times S, ad(P) \otimes \Omega_{X \times S/S}^1)$ ,  $\chi(s) = (F_1(s), \dots, F_l(s))$ . Then it follows that  $\mathcal{N}ilp$  is the fiber of  $\chi$  over  $0 \in H$ .

The *global nilpotent cone*  $\mathcal{N}ilp$ , has been shown to be a *Lagrangian* substack of  $T^*Bun_G$  by various authors(see [4], [2]). As explained in [2], it follows from this fact that the Hitchin morphism  $\chi$  is flat and surjective and further  $T^*Bun_G$  is good

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in the sense of Beilinson and Drinfeld [2].

We will study in this note, a natural notion of the *global nilpotent cone* in the case of the cotangent stack of the moduli stack of parahoric torsors as defined in [1]. The more familiar notion of parabolic bundles on curves, is a special case of parahoric torsors. The main results we derive regarding the *global nilpotent cone* for parahoric torsors are the following:

(A) The *global nilpotent cone* for parahoric torsors is an isotropic substack of the cotangent stack of the moduli stack of parahoric torsors.

(see Section 3, Proposition 1).

(B) We consider a natural analogue of the Hitchin morphism in the case of parahoric torsors and conclude from Proposition 1, that if the dimension of the image of the Hitchin morphism is less than or equal to that of the dimension of the moduli stack of torsors, then the *global nilpotent cone* is in fact Lagrangian.

(C) We prove in the case of the symplectic group  $Sp_{2n}$ , that the Hitchin morphism factors through a subvariety of dimension the same as that of the moduli stack and hence the *global nilpotent cone* is Lagrangian.

The case of parabolic bundles with full flags, has been previously studied by Faltings [3] and the *global nilpotent cone* was proved to be Lagrangian. The case of  $G = GL_n, SL_n$ , has been studied as well by Peter Scheinost and Martin Schottenloher in [9]. This work is mainly a result of trying to understand the results in [9].

## 2. PARABOLIC BUNDLES

Let us assume from now on that  $G$  is simple and simply-connected.

Recall (see [10]) a parabolic  $G$ -bundle on  $(X, x)$  of weight  $\theta$ , is a triple  $(E, \phi, \theta)$ , where

- (i)  $E$  is a principal  $G$ -bundle.
- (ii)  $\theta \in \mathfrak{U}^\circ := \{\theta \in Y(T) \otimes \mathbb{Q} \mid (\theta, \alpha_i) \geq 0, (\theta, \alpha_0) < 1\}$
- (iii)  $\phi$  is a reduction of structure group of  $E$  at  $x$  to the parabolic subgroup  $P_\theta \subset G$ , determined by  $\theta$ .

It was shown by Balaji and Seshadri ([1]), that one can equivalently think of parabolic  $G$ -bundles on  $(X, x)$ , as torsors under certain group schemes which are called Bruhat-Tits group schemes. Let us briefly recall the results from [1]:

Let  $\theta \in \mathfrak{U}$  and for any root  $r \in R$ , consider the integer

$$m_r(\theta) = -\lfloor (\theta, r) \rfloor$$

where  $(, ) : Y(T) \times X(T) \rightarrow \mathbb{Z}$ , is the usual pairing.

Identify the completed local ring  $A = \hat{\mathcal{O}}_{X,x}$ , with the power series ring  $\mathbb{C}[[z]]$ . We denote by  $K$ , the fraction field of  $A$ . Then the subgroup

$$\mathcal{P}_\theta := \langle T(A), z^{m_r(\theta)} U_r(A) \rangle \subset G(K)$$

is a parahoric subgroup in the sense of Bruhat-Tits. Thus from Bruhat-Tits theory, we get a smooth affine group scheme  $\mathcal{G}_\theta$  over  $\text{spec}(A)$ , which satisfies:

- (i)  $\mathcal{G}_\theta \times_{\text{spec}(A)} \text{Spec}(K) \cong G \times \text{spec}(K)$ .
- (ii)  $\mathcal{G}_\theta(A) = \mathcal{P}_\theta$ .

One can construct a smooth affine group scheme  $\mathcal{G}_{X,x,\theta}$  on  $X$  such that:

- (i)  $\mathcal{G}_{X,x,\theta} \mid_{X-x} \cong G \times (X - x)$ .
- (ii)  $\mathcal{G}_{X,x,\theta} \mid_{\text{spec}(A)} \cong \mathcal{G}_\theta$ . In [1], it is shown that for  $\theta \in \mathfrak{U}^\circ$ , we have

$$\mathcal{P}_\theta = \text{ev}^{-1}(P_\theta).$$

where  $ev : G(A) \rightarrow G$  is the natural map and  $P_\theta$  is the parabolic subgroup determined by  $\theta$ . Further in this case, a  $\mathcal{G}_{X,x,\theta}$  torsor on  $(X, x)$  is the same as a parabolic  $G$ -bundle of weight  $\theta$ .

For the rest of this article, a parabolic  $G$ -bundle of weight  $\theta$  on  $(X, x)$  is a  $\mathcal{G}_{X,x,\theta}$  torsor on  $(X, x)$ . We will denote by  $Bun_{\mathcal{G}_{X,x,\theta}}$ , the moduli stack of parabolic  $G$  bundles of weight  $\theta$  on  $(X, x)$ .

### 3. GLOBAL NILPOTENT CONE OF $T^*Bun_{\mathcal{G}_{X,x,\theta}}$

We have  $Bun_{\mathcal{G}_{X,x,\theta}}$  is a smooth equidimensional algebraic stack. Let  $S$  be a scheme over  $\text{spec}(\mathbb{C})$ . From standard deformation theory, we have

$$T^*Bun_{\mathcal{G}_{X,x,\theta}}(S) = \{(E, s) \mid E \in Bun_{\mathcal{G}_{X,x,\theta}}(S), s \in H^0(X \times S, ad(E)^* \otimes \Omega_{X \times S/S}^1)\}.$$

Since  $\mathcal{G}_{X,x,\theta} \mid_{X-x} \cong G \times (x-x)$ , we have, via the isomorphism  $\mathfrak{g} \cong \mathfrak{g}^\vee$  induced by the killing form,  $ad(E)^* \otimes \Omega_{X \times S/S}^1 \mid_{X-x \times S} \cong ad(E) \otimes \Omega_{X \times S/S}^1 \mid_{X-x \times S}$ . Thus for any point  $y \in X - x \times S$ , we have

$$s(y) \in \mathfrak{g} \otimes k(y).$$

Thus we say  $s \mid_{X-x \times S}$  is nilpotent if

$$\forall y \in X - x \times S, \quad s(y) \in \mathfrak{g} \otimes k(y) \text{ is nilpotent.}$$

We define the *global nilpotent cone* as the substack  $\mathcal{N}ilp_{X,x,\theta}$ , defined as

$$\mathcal{N}ilp_{X,x,\theta}(S) = \{(E, s) \in T^*Bun_{\mathcal{G}_{X,x,\theta}} \mid s \mid_{X-x \times S} \text{ is nilpotent}\}.$$

We then have the following

**Proposition 1.**  $\mathcal{N}ilp_{X,x,\theta}$  is an isotropic substack of  $T^*Bun_{\mathcal{G}_{X,x,\theta}}$ .

*Proof.* The proof is an adaptation of [4, Lemma 5, pg 516] and hence we will contend ourselves by explaining the necessary modifications involved. Let  $E$  be a  $\mathcal{G}_{X,x,\theta}$  torsor on  $(X, x)$  and  $s \in H^0(X, ad(E)^* \otimes K_X)$  be nilpotent. Let  $\mathcal{B} \subset \mathcal{G}_{X,x,\theta}$  be the borel subgroup scheme, defined as the flat closure of  $X - x \times B$  in  $\mathcal{G}_{X,x,\theta}$ , for a borel subgroup  $B \subset G$ . From Heinloth [5][Lemma 23], we have the natural morphism  $f : Bun_{\mathcal{B}} \rightarrow Bun_{\mathcal{G}_{X,x,\theta}}$  is surjective.

Now we can find a finite Galois cover  $p : Y \rightarrow X$ , with Galois group  $\Gamma$ , such that the stack of  $\Gamma$  equivariant principal  $G$ -bundles on  $Y$  of a fixed local type determined by  $\theta$ , is equivalent to the stack of  $\mathcal{G}_{X,x,\theta}$  torsors on  $X$ . For a choice of such an equivariant bundle  $F$  on  $Y$ , we have

$$\mathfrak{R}_{Y/X}(Ad(F))^\Gamma \cong \mathcal{G}_{X,x,\theta}$$

where  $\mathfrak{R}_{Y/X}()$  denotes the restriction of scalars functor. The equivalence of the categories of equivariant bundles on  $Y$  and  $\mathcal{G}_{X,x,\theta}$  is obtained by

$$N \mapsto \mathfrak{R}_{Y/X}^\Gamma(N \wedge F^{op})$$

Further we have a  $\Gamma$  equivariant  $B$  reduction  $F_B$  of  $F$ , such that  $\mathfrak{R}_{Y/X}^\Gamma(Ad(F_B)) \cong \mathcal{B}$ . As in [4], choose a  $B$  reduction of  $F$  over the generic point of  $X$ , so that we have  $s \in \mathfrak{n}_F \otimes K_X$ . If  $N$  is the equivariant bundle on  $Y$ , which corresponds to  $F$  under the equivalence mentioned above, then we have an equivariant section  $\tilde{s}$  of  $ad(N) \otimes K_Y$ , which descends to  $s$ . Further we get an equivariant  $B$  reduction of  $N$  over the generic point of  $Y$ , so that  $\tilde{s}$  is a section of  $\mathfrak{n}_N \otimes K_Y$ . Since  $G/B$  is projective, we can extend this  $B$  reduction  $N_B$  of  $N$  to the whole of  $Y$ . Thus we have a  $\mathcal{B}$  reduction of  $F$  given by

$$F_B = \mathfrak{R}_{Y/X}^\Gamma(N_B \wedge E_B)$$

such that over the generic point,  $s \in \mathfrak{n}_F \otimes K_X$ . Thus we have  $f^*(s) \in H^0(X, ad(F_{\mathcal{B}}) \otimes K_X) = 0$  as it vanishes on the generic point. Following the notations of [4, Lemma 3], for  $N_1 = Bun_{\mathcal{B}}$  and  $N_2 = Bun_{\mathcal{G}_{X,x,\theta}}$ , we have  $Nilp_{X,x,\theta} = pr_2(Y_f)$ . The rest of the proof can be done exactly as in the proof of [4][Lemma 5].  $\square$

**Remark 1.** *In the above Proposition, we have nowhere used the fact that  $\theta$  lies in the interior of the weyl alcove. Hence the above Lemma holds true for general parahoric torsors as considered in [1].*

#### 4. HITCHIN MAP FOR PARAHORIC TORSORS

Let  $\widehat{\mathfrak{p}} \subset \mathfrak{g}((z))$  be a parahoric subalgebra. Let  $\kappa(\cdot, \cdot)$  be the killing form on  $\mathfrak{g}$ . We define the *dual* of  $\widehat{\mathfrak{p}}$  as

$$\widehat{\mathfrak{p}}^\vee = \{u \in \mathfrak{g}((z)) \mid \kappa(u, v) \in \mathbb{C}[[t]], \forall v \in \widehat{\mathfrak{p}}\}$$

We have a natural isomorphism

$$\Psi : \widehat{\mathfrak{p}}^\vee \rightarrow Hom_{\mathbb{C}[[t]]}(\widehat{\mathfrak{p}}, \mathbb{C}[[t]])$$

given by

$$\Psi(u) := \Psi(u)(v) = \kappa(u, v), \quad \forall v \in \widehat{\mathfrak{p}}, \quad u \in \widehat{\mathfrak{p}}^\vee.$$

Hence it makes sense to call  $\widehat{\mathfrak{p}}^\vee$ , the *dual* of  $\widehat{\mathfrak{p}}$ .

Consider now the case when  $\widehat{\mathfrak{p}} = ev^{-1}(\mathfrak{p})$  for a parabolic sub-algebra  $\mathfrak{p} \subset \mathfrak{g}$ . We then have the following lemma

**Lemma 1.** *Let  $\widehat{\mathfrak{p}} = ev^{-1}(\mathfrak{p})$  for a parabolic subalgebra  $\mathfrak{p} \subset \mathfrak{g}$ . We then have a natural equality of lattices in  $\mathfrak{g}((z))$*

$$\frac{1}{z}(\mathfrak{n} + z\mathfrak{g}[[z]]) = \widehat{\mathfrak{p}}^\vee.$$

where  $\mathfrak{n}$  is the nil-radical of  $\mathfrak{p}$ .

*Proof.* Let  $\mathfrak{t} \subset \mathfrak{p}$  be a cartan subalgebra. Let  $R = R^+ \cup R^-$  be the set of roots with respect to  $\mathfrak{t}$ . We then have the familiar decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\beta \in R} \mathfrak{g}_\beta$$

Let  $\{x_\beta \in \mathfrak{g}_\beta\}_{\beta \in R} \cup \{h_1, \dots, h_l \mid h_i \in \mathfrak{t}\}$  be a basis of  $\mathfrak{g}$ . Using the killing form  $\kappa$ , we can identify  $\mathfrak{g}$  and  $\mathfrak{g}^\vee$ . Consider now the linear function  $f_\beta$  for  $\beta \in R$ , given by

$$f_\beta(g) = \begin{cases} 0 & g \in \mathfrak{t} \\ 0 & g \in \mathfrak{g}_\alpha, \alpha \neq \beta. \\ c & g = cx_\beta. \end{cases}$$

We then have under the isomorphism  $\mathfrak{g} \cong \mathfrak{g}^\vee$ , via the killing form, the linear function  $f_\beta$  corresponds to  $x_{-\beta}$ . Similarly we can define  $f_i$  by

$$f_i(g) = \begin{cases} 0 & g \notin \mathfrak{t} \\ 0 & g \in \mathfrak{t}, \quad g \in \langle h_j \rangle_{j \neq i} \\ c & g = ch_i. \end{cases}$$

We then have  $f_i$  corresponds to an element in  $\mathfrak{t}$  under the isomorphism  $\mathfrak{g} \cong \mathfrak{g}^\vee$ . Now let  $\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\beta \in R^+} \mathfrak{g}_\beta \oplus \bigoplus_{\alpha \in R^* \subset R^+} \mathfrak{g}_{-\alpha}$  and  $\mathfrak{n} = \bigoplus_{\delta \in R^*} \mathfrak{g}_\delta$ .

We then have

$$\widehat{\mathfrak{p}} = \mathfrak{t}[[z]] \oplus \bigoplus_{\beta \in R^+} \mathfrak{g}_\beta \otimes \mathbb{C}[[z]] \oplus \bigoplus_{\alpha \in R^* \subset R^+} \mathfrak{g}_{-\alpha} \otimes \mathbb{C}[[z]] \oplus \bigoplus_{\delta \notin R^*} \mathfrak{g}_{-\delta} \otimes z\mathbb{C}[[z]]$$

Hence

$$\widehat{\mathfrak{p}}^\vee = \langle f_i, \{f_\beta\}_{\beta \in R^+}, \{f_{-\alpha}\}_{\alpha \in R^*}, \{\frac{1}{z}f_{-\delta}\}_{\delta \notin R^*} \rangle \subset \mathbb{C}[[z]].$$

Thus we get from the above discussion,

$$\begin{aligned} \widehat{\mathfrak{p}}^\vee &= \langle \mathfrak{t}, \{\mathfrak{g}_\beta\}_{\beta \notin R^*}, \{\frac{1}{z}\mathfrak{g}_\delta\}_{\delta \in R^*} \rangle \subset \mathbb{C}[[z]] \\ &= \frac{1}{z}(\mathfrak{n} + z\mathfrak{g}[[z]]) \end{aligned}$$

□

Let  $E$  be a  $\mathcal{G}_{X,x,\theta}$  torsor on  $X$ . Let  $\mathfrak{p} = \text{Lie}(\mathcal{P}_\theta) \subset \mathfrak{g}((z))$ , which is a parahoric subalgebra of  $\mathfrak{g}((z))$ . We then have an identification of lattices in  $\mathfrak{g}((z))$

$$\widehat{ad(E)_x}^* = \mathfrak{p}^\vee$$

where  $\widehat{ad(E)_x}^*$  is the completion of the stalk of  $ad(E)^*$  at  $x$ . Now choose a homogeneous generating set  $\{F_1, \dots, F_l\}$  for  $\mathbb{C}[\mathfrak{g}]^G$ . Since the set  $\widehat{\mathfrak{p}}^\vee$  is a bounded subset of  $\mathfrak{g}((z))$ , we have for  $C \gg 0 \in \mathbb{N}$ ,

$$\nu_t(F_i(\alpha)) \geq -C, \quad \forall 1 \leq i \leq l, \quad \forall \alpha \in \mathfrak{p}^\vee.$$

Thus we can define a Hitchin morphism

$$\chi : \text{Bun}_{\mathcal{G}_{X,x,\theta}} \rightarrow \bigoplus_{i=1}^l H^0(X, K_X^{\otimes d_i}(Cx))$$

An immediate corollary of Proposition 1 is the following:

**Corollary 1.** *If we can find a subvariety  $B \subset \bigoplus_{i=1}^l H^0(X, K_X^{\otimes d_i}(Cx))$ , such that  $\dim(B) = \dim(\text{Bun}_{\mathcal{G}_{X,x,\theta}})$  and  $\chi$  factors through  $B$ , then we have*

(i)  $\dim(\mathcal{N}ilp_{X,x,\theta}) = \dim(\text{Bun}_{\mathcal{G}_{X,x,\theta}})$

(ii)  $\chi$  is flat and surjective.

(iii)  $\mathcal{N}ilp_{X,x,\theta}$  is a Lagrangian substack of  $T^*\text{Bun}_{\mathcal{G}_{X,x,\theta}}$ .

*sketch of the proof.* We only give a sketch of the proof, since we just have to follow the arguments in [4][Proposition 1, Corollary 9, Theorem 10]. We have  $\dim(T^*(\text{Bun}_{\mathcal{G}_{X,x,\theta}})) \geq 2\dim(\text{Bun}_{\mathcal{G}_{X,x,\theta}})$  as  $\text{Bun}_{\mathcal{G}_{X,x,\theta}}$  is an equidimensional stack (see [2]). The fact that  $\dim(B) = \dim(\text{Bun}_{\mathcal{G}_{X,x,\theta}})$ , tell us that any fiber of  $\pi$  has dimension greater than or equal to  $\dim(T^*(\text{Bun}_{\mathcal{G}_{X,x,\theta}})) - \dim(B) \geq \dim(\text{Bun}_{\mathcal{G}_{X,x,\theta}})$ . But as we have from Proposition 1, that  $\mathcal{N}ilp_{X,x,\theta}$  is isotropic, we must have  $\dim(\mathcal{N}ilp_{X,x,\theta}) = \dim(\text{Bun}_{\mathcal{G}_{X,x,\theta}})$  and  $\mathcal{N}ilp_{X,x,\theta}$  is in fact Lagrangian. We have  $\mathcal{N}ilp_{X,x,\theta}$  is the fiber over 0 of the Hitchin morphism. Using The natural  $\mathbb{C}^\times$  action on the cotangent stack, we can put any fiber of  $\chi$  in a family parametrized by  $\mathbb{A}^1$ , so that the central fiber is a substack of  $\mathcal{N}ilp_{X,x,\theta}$  and all other fibers have the same dimension. Thus we must have the dimension of the general fiber is less than that of the special fiber, which equals  $\dim(\text{Bun}_{\mathcal{G}_{X,x,\theta}})$ . Hence we get

$$\dim(\chi^{-1}(b)) = \dim(\text{Bun}_{\mathcal{G}_{X,x,\theta}}), \quad \forall b \in B.$$

Hence  $\chi$  is flat and surjective.

□

**Corollary 2.** *[case of parabolic bundles with full flag] For  $\theta \in \mathfrak{U}^0$ , such that  $P_\theta$  is a Borel subgroup of  $G$ , we have the global nilpotent cone  $\mathcal{N}ilp_{X,x,\theta}$  is Lagrangian.*

*Proof.* From Corollary 1, it is enough to show that the Hitchin morphism factors through a subvariety of dimension equal to that of  $\text{Bun}_{\mathcal{G}_{X,x,\theta}}$ . Now we have

$$\dim(\text{Bun}_{\mathcal{G}_{X,x,\theta}}) = \dim(G)(g-1) + \dim(G/B).$$

where  $B = P_\theta$ . On the otherhand, let  $\widehat{\mathfrak{b}} = ev^{-1}(Lie(B))$ . Let  $\mathfrak{n}$  be the nil-radical of  $Lie(B)$ . Let  $E$  be a  $\mathcal{G}_{X,x,\theta}$  torsors on  $(X, x)$ . We have from Lemma 1, and the above discussions

$$\widehat{ad(E)}_x^* \simeq \widehat{\mathfrak{p}}^\vee = \frac{1}{z}(\mathfrak{n} + z\mathfrak{g}[[z]]).$$

Let  $\{f_1, \dots, f_l\}$  be homogeneous invariants generators for  $\mathbb{C}[\mathfrak{g}]^G$  and  $deg(f_i) = d_i$ . Recall, we have

$$\sum_i d_i = \dim(G/B) + l.$$

For any  $e \in \mathfrak{n}$ , we have

$$\nu_z(f_i(e + M)) \geq 1, \quad \forall M \in z\mathfrak{g}[[z]], \forall j$$

as  $f_j(e) = 0$ . Thus we get

$$\nu_z(f_i(\frac{1}{z}(e + M))) \geq -(d_i - 1), \quad \forall i, \forall M \in z\mathfrak{g}[[z]]$$

Thus we have the image of the Hitchin morphism is contained in the subspace

$$W = \bigoplus_i H^0(X, K_X^{d_i}(d_i - 1))$$

From Riemann-Roch theorem, we get

$$\dim(W) = \dim(G)(g - 1) + \sum_i (d_i - 1) = \dim(G)(g - 1) + \dim(G/B)$$

□

## 5. SOME LOCAL COMPUTATIONS

In this section, we will study the local picture of the Hitchin morphism in the case of  $\mathfrak{gl}_m$ , for a particular choice of invariant generators, which are the various co-efficients of the characteristic polynomial. The local study helps us conclude that, for a suitable choice of invariant generators for  $\mathfrak{sp}_{2n}$ , the Hitchin morphism for symplectic parabolic bundles, factors through a subvariety of dimension same as that of the moduli stack. Thus from Corollary 1, we get the *global nilpotent cone* is Lagrangian. The combinatorial results and the other statements we derive in this section, are a result of trying to understand the statements found in [9][pages 209-215].

Let  $e \in \mathfrak{gl}_m$  be a nilpotent element. We can associate to the conjugacy class of  $e$ , a partition  $\mu = (m_1, \dots, m_k)$  of  $m$ . We denote by  $\tilde{\mu} = (\tilde{m}_1, \dots, \tilde{m}_r)$ , the dual partition of  $m$ . Now assume  $m = 2n$  and  $e \in \mathfrak{sp}_{2n} \subset \mathfrak{gl}_{2n}$ . It is a well-known fact that we have, the corresponding partition  $\mu$ , satisfies the property

in  $\mu$  every odd number occurs with even multiplicity

Further we have

$$\dim(Z_{SP_{2n}}(e)) = \frac{1}{2}(\sum_i \tilde{m}_i^2 + \#\{i \mid m_i \text{ is odd}\})$$

Now for a number  $1 \leq j \leq m$ , we define

$n(\mu, j) = a$ ; where  $a$  is the unique number such that  $m_1 + \dots + m_{a-1} < j \leq m_1 + \dots + m_a$ .

We denote by  $F_j \in \mathbb{C}[\mathfrak{gl}_m]^{GL_m}$ , defined by

$$\det(Q - TId_{m \times m}) = T^m + F_1(Q)T^{m-1} + \dots + F_m(Q).$$

Following Khazdan and Lusztig ([6]), we say a subset  $Y \subset \mathfrak{g}[[z]]$  is constructible, if for some integer  $N$ , we have a constructible subset  $Y_l$  of the affine space  $\mathfrak{g}[[t]]/t^l \mathfrak{g}[[t]]$ , such that under the natural map

$$f_l : \mathfrak{g}[[t]] \rightarrow \mathfrak{g}[[t]]/t^l \mathfrak{g}[[t]]$$

we have

$$Y = f_l^{-1}(Y_l).$$

A subset  $U \subset Y$ , where  $Y$  is constructible as above is called open, if for some  $l_1 > l$ , we have  $\tilde{U} \subset (p_{l_1}^{l_1})(Y_l)$  open and  $U = p_{l_1}^{-1}(\tilde{U})$ , where

$$p_{l_1}^{l_1} : \mathfrak{g}[[t]]/t^{l_1}\mathfrak{g}[[t]] \rightarrow \mathfrak{g}[[t]]/t^l\mathfrak{g}[[t]].$$

is the natural map. Now we claim the following

**Lemma 2.** *Consider the subset  $\mathcal{I}_e^j \subset e + z\mathfrak{gl}_m[[z]]$ , given by*

$$\mathcal{I}_e^j = \{\gamma \in e + z\mathfrak{gl}_m[[z]] \mid \nu_t(\gamma) \text{ is the minimum}\}$$

*Then  $\mathcal{I}_e^j$  is open in  $e + z\mathfrak{gl}[[z]]$ .*

*Proof.* Let  $k \in \mathbb{N}$ . It is enough to prove the following subsets are open

$$\mathcal{I}_e^{j,k} := \{\gamma \in e + z\mathfrak{gl}[[z]] \mid \nu_t(F_j(\gamma)) \leq k\}$$

To see this, consider the induced morphism of varieties

$$F_j^k : \mathfrak{gl}[[z]]/z^{k+1}\mathfrak{gl}[[z]] \rightarrow \mathbb{C}[[z]]/z^{k+1}\mathbb{C}[[z]].$$

Let  $f_{k+1} : \mathfrak{gl}[[z]] \rightarrow \mathfrak{gl}[[z]]/z^{k+1}\mathfrak{gl}[[z]]$  be the natural map, which is surjective. Now we have

$$F_j^k(f_{k+1}(\gamma)) \neq 0 \iff \nu_t(F_j(\gamma)) \leq k$$

Thus we get

$$\mathcal{I}_e^{j,k} = f_{k+1}^{-1}((f_0^{k+1})^{-1}(\{e\}) \cap (F_j^k)^{-1}(0))$$

and hence is open.  $\square$

We then have following proposition

**Proposition 2.** *Let  $\gamma \in e + z\mathfrak{gl}_m[[z]]$ . Then we have*

$$\nu_t(F_j(\gamma)) \geq n(\mu, j)$$

*Proof.* We can assume with no loss of generality that  $e$  is in the jordan form, with jordan blocks of sizes  $m_i$  along the diagonal. Let  $\mathfrak{l}$  be the sub-algebra of  $\mathfrak{gl}_m$ , consisting of blocks of sizes  $m_i$  along the diagonal. We then have

$$\mathfrak{l} \cong \mathfrak{gl}_{m_1} \times \cdots \times \mathfrak{gl}_{m_r}$$

and  $e \in \mathfrak{l}$  is regular nilpotent in  $\mathfrak{l}$ . Let  $e = e_1 + e_2 + \cdots + e_r$  be the jordan decomposition, and  $\{F_{ij}\}_{j=1}^{m_i}$  be the invariants in  $\mathbb{C}[\mathfrak{gl}_{m_i}]^{GL_{m_i}}$  given by the co-efficients of characteristic polynomials. For  $M \in \mathfrak{l}$ , we think of  $M$  as a tuple  $(M_1, \dots, M_r)$ , where  $M_i \in \mathfrak{gl}_{m_i}$ . Using the formula

$$\det(M - TId) = \det(M_1 - TId) \cdots \det(M_r - TId)$$

we can express  $F_j|_{\mathfrak{l}}$ , as a sum of products of the form  $F_{i_1 j_1} \cdots F_{i_h j_h}$ , where  $i_1 \neq i_2 \cdots \neq i_h$  and  $j_1 + \cdots + j_h = j$ . The minimum number of such terms possible is  $\mu(j)$  and thus we get

$$\nu_t(F_j(\gamma)) \geq n(\mu, j), \quad \gamma \in e + z\mathfrak{l}[[z]]$$

since we have  $\nu_t(F_{ij}(\gamma)) \geq 1$ , for  $\gamma \in e + z\mathfrak{l}[[z]]$ . Now we consider the open set of regular semisimple elements in  $e + z\mathfrak{gl}[[z]]$  as considered by Khazdan and Lusztig ([6][pg 156]) which over  $\overline{\mathbb{C}((t))}$  is conjugate to a matrix of the form  $D = \text{diag}(a_1^1, \dots, a_{m_1}^1, a_1^2, \dots, a_1^r, \dots, a_{m_r}^r)$ , where

$$a_\lambda^i \in z^{\frac{1}{m_i}} \mathbb{C}[[z^{\frac{1}{m_i}}]]$$

We will denote this open set by  $\mathcal{J}_e$ . Observe we have  $D \in \mathfrak{l} \otimes \overline{\mathbb{C}((t))}$  and  $F_{ij}(D) \in z\mathbb{C}[[z]]$ ,  $\forall i, j$ . Choose an  $\mathfrak{sl}_2$  triple  $\{e, f, h\} \subset \mathfrak{l}$ . Let  $V = \mathfrak{z}_{\mathfrak{l}}(f)$ . Choose a basis  $\{v_{ij}\}$

of  $V$  consisting of eigenvectors for  $t$ . Now from a theorem of Kostant([7]), we can find generators  $\{Q_{ij}\}$  for the ring  $\mathbb{C}[[\mathfrak{l}]]^L$ , such that we have

$$Q_{ij}(e + \Sigma a_{pq} v_{pq}) = a_{ij}.$$

Since we have  $\{F_{ij}\}$  are generators for the invariant ring  $\mathbb{C}[[\mathfrak{l}]]^L$  as well, we get

$$Q_{ij}(D) = d_{ij} \in z\mathbb{C}[[z]].$$

Thus we get

$$Q_{ij}(D) = Q_{ij}(e + \Sigma d_{pq} v_{pq}).$$

consequently we get

$$F_j(\tilde{D}) = F_j(D) = F_j(e + \Sigma d_{pq} v_{pq}),$$

where  $\tilde{D} \in \mathcal{J}_e$  is conjugate to  $D$  over  $\overline{\mathbb{C}((t))}$ . In particular, we get

$$\nu_t(F_j(\tilde{D})) \geq n(\mu, j), \quad \tilde{D} \in \mathcal{J}_e.$$

Now to finish off the proof consider the open set  $\mathcal{I}_e^j$  as in Lemma 2. Since  $e\zeta\mathfrak{g}[[z]]$  is an irreducible contructible subset of  $\mathfrak{g}[[z]]$ , we have

$$\mathcal{J}_e \cap \mathcal{I}_e^j \neq \emptyset.$$

Thus we get the minimum of  $\nu_t(F_j)$  attained in  $e + z\mathfrak{g}[[z]]$  is atleast  $\mu(j)$  and hence the lemma is proved.  $\square$

**Lemma 3** (A combinatorial identity). *We have*

$$\Sigma_i i m_i = \frac{1}{2}(\Sigma_j \tilde{m}_j^2 + \tilde{m}_j)$$

*Proof.* Consider the young tableau corresponding to  $\mu$ . Enter the number  $i$  on every box in the  $i^{th}$  row of the tableau. Now summing the resulting set of numbers row-wise, we get the number  $\Sigma_i i m_i$ . While summing up the numbers, column wise, we get the number  $\frac{1}{2}(\Sigma_j \tilde{m}_j^2 + \tilde{m}_j)$ . Thus we get the required equality

$$\Sigma_i i m_i = \frac{1}{2}(\Sigma_j \tilde{m}_j^2 + \tilde{m}_j).$$

$\square$

For  $\mathfrak{g} = \mathfrak{sp}_{2n}$ , the polynomial functions  $\{F_{2i} \mid_{\mathfrak{sp}_{2n}}\}_{i=1}^n$  is a minimal homogeneous generating set for the ring  $\mathbb{C}[\mathfrak{sp}_{2n}]^{Sp_{2n}}$ . We have the following corollary of the Lemma 3

**Corollary 3.** *Let  $\mu$  be the partition corresponding to a nilpotent element  $e \in \mathfrak{sp}_{2n} \subset \mathfrak{gl}_{2n}$ . Then we have*

$$\Sigma_{j=1}^n n(\mu, 2j) = \frac{n}{2} + \frac{1}{2}(\dim(Z_{Sp_{2n}}(e)))$$

*Proof.* We have

$$n(\mu, 2j) = a, \quad m_1 + \cdots + m_{a-1} < 2j \leq m_1 + \cdots + m_a.$$

Thus we have

$$\lfloor m_1 + \cdots + m_{a-1} \rfloor + 1 \leq j \leq \lfloor m_1 + \cdots + m_a \rfloor.$$

Thus consider the subset of  $C_a \subset \{1, \dots, n\}$ , given by

$$C_a = \{j \mid \mu(j) = a\}$$

We have

$$\#C_a = \begin{cases} \frac{m_a}{2} & \text{if } m_a \text{ is even} \\ \frac{m_a-1}{2} & \text{if } m_a \text{ is odd and } m_1 + \cdots + m_{a-1} \text{ is even} \\ \frac{m_a+1}{2} & \text{if } m_a \text{ is odd and } m_1 + \cdots + m_{a-1} \text{ is odd} \end{cases}$$



We have now

$$\Sigma_{j=1}^n n(\mu, 2j) = \Sigma_{a=1}^r a \# C_a.$$

Now since  $\mu$  corresponds to a nilpotent orbit in  $\mathfrak{sp}_{2n}$ , recall we have odd numbers occurs with even multiplicity in  $\mu$ . Now consider the set  $C^o = \{a \mid m_a \text{ is odd}\}$ . Arrange the numbers in  $C^o$  in increasing order and let them be  $\{a_1, \dots, a_{2q}\}$ . We then have

$$a_i = a_{i+1} + 1, \quad i = 1, 3, \dots, 2q - 1.$$

Thus we get that

$$\begin{aligned} \Sigma_{a \in C^o} a \# C_a &= \Sigma_{a \in C^o} \frac{am_a}{2} + \frac{1}{2}(-a_1 + a_2 - a_3 + \dots) \\ &= \Sigma_{a \in C^o} \frac{am_a}{2} + \frac{1}{2}(\frac{1}{2} \# C^o) \end{aligned}$$

□

Thus we have

$$\begin{aligned} \Sigma_{j=1}^n \mu(2j) &= \Sigma_{a \notin C^o} a \# C_a + \Sigma_{a \in C^o} a \# C_a. \\ &= \frac{1}{2}(\Sigma_a am_a + \frac{1}{2} \# C^o) \\ &= \frac{1}{2}(\frac{1}{2}(\Sigma_k \tilde{m}_k^2 + \# C^o) + n) \\ &= \frac{n}{2} + \frac{1}{2}(\dim(Z_{Sp_{2n}}(e))) \end{aligned}$$

**Remark 2.** For  $\mathfrak{g} = \mathfrak{so}_{2n+1} \subset \mathfrak{gl}_{2n+1}$ . We have  $\{F_{2i} \mid \mathfrak{g}\}_{i=1}^n$  is a generating set for  $\mathbb{C}[\mathfrak{g}]^{So_{2n+1}}$ . But the analogue of Corollary 3 fails in this case, as can be seen by the following example:

Let  $\mathfrak{g} = \mathfrak{so}(5)$  and  $e \in \mathfrak{g}$ , whose associated partition of 5 is  $\mu = (2, 2, 1)$ . The dual partition of  $\mu$  is  $(3, 2)$ . We have two invariants  $F_2$  and  $F_4$  and

$$n(\mu, 2) + n(\mu, 4) = 1 + 2 = 3.$$

But on the otherhand, we have

$$\frac{1}{2}(n + \dim(Z_{\mathfrak{g}}(e))) = \frac{1}{2}(2 + 3^2 + 2^2 - 1) = 4.$$

## 6. HITCHIN MORPHISM FOR SYMPLECTIC PARABOLIC BUNDLES

Let  $\mathcal{G}_{X,x,\theta}$  be a Bruhat-Tits group scheme over  $(X, x)$  for the symplectic group  $Sp_{2n}$  and  $\theta$  in the interior of the rational weyl alcove. Let  $P_\theta$  be the parabolic subgroup of  $Sp_{2n}$ , determined by  $\theta$ . Denote by  $\mathfrak{p}_\theta$ , the lie algebra of  $P_\theta$ . Let  $\mathfrak{n}_\theta$  be the nil radical of  $\mathfrak{p}_\theta$ .

We then have

$$\widehat{\mathfrak{p}}_\theta := \widehat{Lie(\mathcal{G}_{X,x,\theta})} \mid_{x=ev^{-1}(\mathfrak{p}_\theta)}$$

where we identify the completed local ring  $\widehat{\mathcal{O}}_x$  with  $\mathbb{C}[[z]]$  and  $ev : \mathfrak{sp}_{2n}[[z]] \rightarrow \mathfrak{sp}_{2n}$  is the natural map. From Lemma we have

$$\widehat{\mathfrak{p}}_\theta^\vee = \frac{1}{z}(\mathfrak{n}_\theta + z\mathfrak{g}[[z]])$$

As in the previous section, we denote by  $F_j \in \mathbb{C}[\mathfrak{sp}_{2n}]^{Sp_{2n}}$ , the invariants defined by the equation

$$\det(M - TId) = \Sigma_{j=0}^{2n} F_j(M) T^{2n-j}$$

It is a well-known fact that, we have  $F_j = 0$ , for  $j$  odd and  $\{F_{2j}\}_{j=1}^n$  generates the algebra of invariants. Let  $O_\theta \in \mathfrak{n}_\theta$ , be the dense open  $P_\theta$  orbit corresponding to the Richardson class. Let the corresponding partition of  $2n$  be  $\Lambda = (\lambda_1, \dots, \lambda_r)$ .

**Lemma 4.** *We have*

$$\nu_t(F_{2j}(\gamma)) \geq -(2j - n(\Lambda, 2j)), \quad \forall \gamma \in \widehat{\mathfrak{p}}_\theta^\vee.$$

*Proof.* Let  $\gamma \in \widehat{\mathfrak{p}}$ . We then have

$$\gamma = \frac{1}{z}(e + z\tilde{\gamma}), \quad e \in \mathfrak{n}_\theta$$

Thus we have

$$\nu_z(F_{2j}(\gamma)) = -2j + \nu_z(F_{2j}(e + z\tilde{\gamma}))$$

Let  $\mu = (m_1, \dots, m_s)$  be the partition of  $2n$  corresponding to  $e$ . We have from Proposition ??, that

$$\nu_z(F_{2j}(\gamma)) \geq -2j + n(\mu, 2j) = -(2j - n(\mu, 2j))$$

Now since  $e \in \overline{O}_\theta$ , we have for any  $k$ ,

$$\sum_{i=1}^k m_i \leq \sum_{i=1}^k \lambda_i.$$

Thus we see that

$$n(\mu, 2j) \geq n(\Lambda, 2j).$$

Hence we get

$$\nu_z(F_{2j}(\gamma)) \geq -(2j - n(\mu, 2j)) \geq -(2j - n(\Lambda, 2j))$$

□

**Corollary 4.** *Let  $C \gg 0$  be such that, we have*

$$\nu_z(F_{2j}(\gamma)) \geq -C, \quad \forall \gamma \in \widehat{\mathfrak{p}}_\theta, \quad j = 1, 2, \dots, n.$$

*Then image of the Hitchin morphism*

$$\chi : T^*Bun_{\mathcal{G}_{X,x,\theta}} \rightarrow \bigoplus_{j=1}^n \bigoplus H^0(X, K_X^{2j}(Cx))$$

*factors through a subvariety of dimension equal to that of  $Bun_{\mathcal{G}_{X,x,\theta}}$ .*

*Proof.* From Lemma 4, we have  $C \leq \min\{2j - n(\Lambda, 2j)\}_{j=1}^n$  and the image of  $\chi$ , lands inside the subspace

$$W = \bigoplus_{j=1}^n H^0(X, K_X^{2j}(2j - n(\Lambda, 2j)))$$

We have from Riemann-Roch formula,

$$\begin{aligned} \dim(W) &= \dim(Sp_{2n})(g-1) + \sum_{j=1}^n (2j - n(\Lambda, 2j)) \\ &= \dim(Sp_{2n})(g-1) + \frac{1}{2}(\dim(Sp_{2n}) - \dim(Z_{Sp_{2n}}(e^\circ))), \quad e^\circ \in O_\theta \\ &= \dim(Sp_{2n})(g-1) + \dim(Sp_{2n}/P_\theta) \\ &= \dim(Bun_{\mathcal{G}_{X,x,\theta}}) \end{aligned}$$

□

**Corollary 5.** *For  $G = Sp_{2n}$  and  $\theta$  an element in the interior of the rational weyl alcove, we have*

- (i)  $\dim(\mathcal{N}ilp_{X,x,\theta}) = \dim(Bun_{\mathcal{G}_{X,x,\theta}})$
- (ii) *The Hitchin morphism  $\chi$  is flat. Further every irreducible component of the fiber of  $\chi$  has dimension equal to that of  $Bun_{\mathcal{G}_{X,x,\theta}}$ .*
- (iii)  $T^*(Bun_{\mathcal{G}_{X,x,\theta}})$  *is a local complete intersection and  $\mathcal{N}ilp_{X,x,\theta}$  is a Lagrangian complete intersection in  $T^*(Bun_{\mathcal{G}_{X,x,\theta}})$ .*

*Proof.* Proof follows exactly as in the proof of Corollary 1

□

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